Final revision "1 sec"

First : Algebra

- $A_{m \times n}$ ⇒ اقصى ordered 3×3 في المنهج

raws ↲ ↳ colums

- Some Special Matrices:

① row Matrics $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ ، $\begin{pmatrix} 2 & -1 \end{pmatrix}$

② Colum Matrics $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ، $\begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$

③ square Matrics $\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 5 & 0 & 8 \\ 1 & 7 & 8 \end{pmatrix}$

④ Zero Matrics $\square$, O $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ كل العناصر Zero

⑤ Diagonal Matrics $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ ، $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}$

⑥ unit Matrics او identity Matrics $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- $(A^t)^t = A$ • $(A+B)^t = A^t + B^t$ • $(AB)^t = B^t A^t$ "مهم"
- if $A = A^t$ is Called a symmetric Matrix
- if $A = -A^t$ is Called skaw symmetric Matrix

يبقى خلي بالك قطرها لازم diagonal يكون ب Zero مهم

- $A^{-1} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Rightarrow \Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- $AA^{-1} = A^{-1}A = I$ "مهم"
- area of ΔABC where $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$

$$\text{area} = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

ولو عاوز اثبت انهم على one strit line يبقى حيبقى

Zero = area

- $\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & F \end{vmatrix} = adF$ • $\begin{vmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & F \end{vmatrix} = adF$

- Solve $a_1x + b_1y = c_1$, $a_2x + b_2y = c_2$ by cramer

$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $\Delta x \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, $\Delta y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$\therefore x = \frac{\Delta x}{\Delta}$, $y = \frac{\Delta y}{\Delta}$

- $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ بكره نوضح طريقه Matricx

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

Second: Trigonometry

- $\sin\theta = \frac{opp}{hyp}$ • $\sin\theta \times \csc\theta = 1$
- $\cos\theta = \frac{adj}{hyp}$ • $\cos\theta \times \sec\theta = 1$
- $\tan\theta = \frac{opp}{adj}$ • $\tan\theta \cdot \cot\theta = 1$
- $x^2 + y^2 = 1$

$\boxed{\cos^2\theta + \sin^2\theta = 1} \Rightarrow ①$

$\cos^2\theta = 1 - \sin^2\theta$,

$\sin^2\theta = 1 - \cos^2\theta$.

$\div \cos^2\theta$

$\boxed{1 + \tan^2\theta = \sec^2} \Rightarrow ②$

$1 = \sec^2\theta - \tan^2\theta$

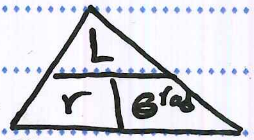
$\tan^2\theta = \sec^2\theta - 1$

$\div \sin^2\theta$

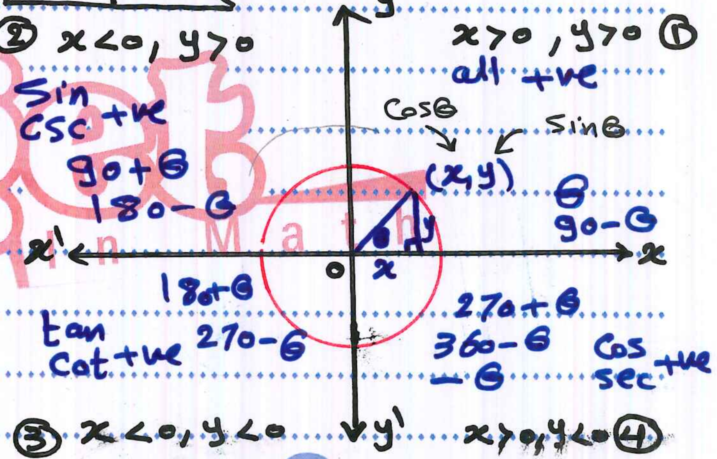
$\boxed{\cot^2\theta + 1 = \csc^2\theta} \Rightarrow ③$

$\cot^2\theta = \csc^2\theta - 1$

$1 = \csc^2\theta - \cot^2\theta$



$$\frac{\theta^\circ}{180} = \frac{\theta^{rad}}{\pi}$$



- 90, 270 Change استرار
- 180, 360 not chenge طبيعي

مع مراعاه اشاره الربع

- if $\sin\theta = \cos\alpha$

$\therefore \theta \pm \alpha = 90 + 360n$

- If $\csc\theta = \sec\alpha$

$\therefore \theta \pm \alpha = 90 + 360n$

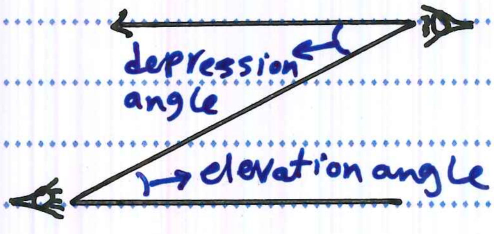
- If $\tan\theta = \cot\alpha$

$\therefore \theta + \alpha = 90 + 180n$

- $-1 \leq \begin{matrix} \sin\theta \\ \cos\theta \end{matrix} \leq 1 \Leftarrow$ مهم جدا

- area of $\Delta = \begin{cases} \frac{1}{2} b \cdot h \\ \frac{1}{2} \text{Two side} \times \text{Sine angle in clouded} \\ \sqrt{\frac{1}{2}P(\frac{1}{2}P-a)(\frac{1}{2}P-b)(\frac{1}{2}P-c)} \end{cases}$

$\left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$



Target In Math
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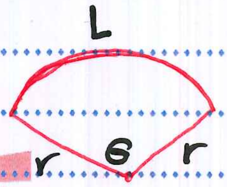
- area of any Quadrilateral $= \frac{1}{2}$ Product of Two diagonals × Sine angle between Them.
- area of any Figure reguler $= \frac{1}{4} n x^2 \cot \frac{\pi}{n}$
  number of side ⟵ ⟶ Length of side
- Circle $\begin{cases} \text{Perimeter} = 2\pi r \\ \text{area} = \pi r^2 \end{cases}$
- The Circular sector

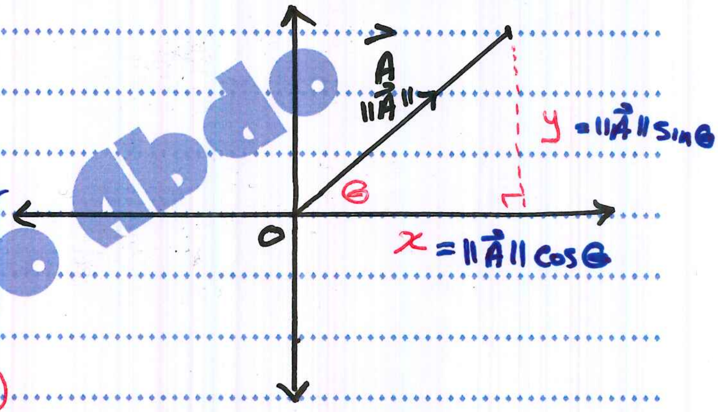


(1) Perimeter $= 2r + L$

(2) Area $\begin{cases} \frac{1}{2} L r \\ \frac{1}{2} \theta^{rad} r^2 \\ \frac{\pi \theta^{\circ} r^2}{360} \end{cases}$

- area of circular segment $= \frac{1}{2} r^2 (\theta^{rad} - \sin\theta)$

Third: Geometry



- $\overrightarrow{OA} = (x, y)$ Position vector

$\|\overrightarrow{OA}\| = \|\vec{A}\| = \sqrt{x^2 + y^2}$

$\tan\theta = \frac{y}{x}$ مع مراعاه اشاره ربع (direction)

$\overrightarrow{OA} = x\vec{i} + y\vec{j}$ (Fundamental)

$\overrightarrow{OA} = (\|\vec{A}\|, \theta^{\circ})$ (Polar vector).

↳ $(\|\vec{A}\| \cos\theta, \|\vec{A}\| \sin\theta)$ Cartizen.

- if $\overrightarrow{AB}$ equivalent $\overrightarrow{xy}$ $\therefore \|\overrightarrow{AB}\| = \|\overrightarrow{xy}\|$

$\therefore \overrightarrow{AB} = \overrightarrow{xy}$ $\therefore B - A = y - x$ and Same direction

خلي بالك unit vector يعني $\| \ \| = 1$

- $\vec{A} \parallel \vec{B}$, $\vec{A}(x_1, y_1)$ $\vec{B}(x_2, y_2)$ $\vec{A} \perp \vec{B}$

(1) $\frac{x_1}{x_2} = \frac{y_1}{y_2}$ $x_1 x_2 + y_1 y_2 = 0$ ⟵ مهم جدا

(2) $x_1 y_2 - x_2 y_1 = 0$

(3) $\vec{A} = k\vec{B}$ ⟶ $\begin{matrix} + \text{ Same} \\ - \text{ opp} \end{matrix}$

Target In Math
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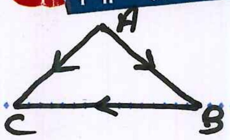
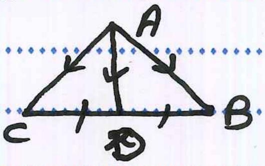
- Add vectors : (1)



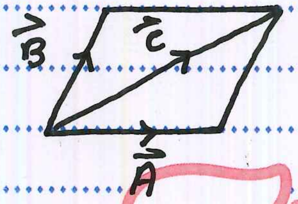
$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

(2)



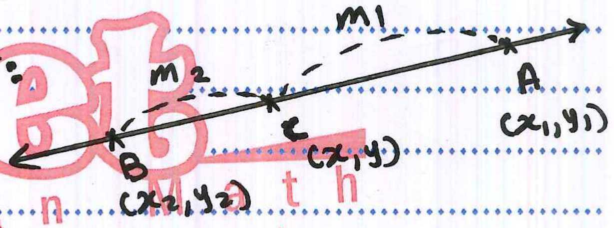
$\overrightarrow{AB}+\overrightarrow{AC}=2\overrightarrow{AD}$

(3)



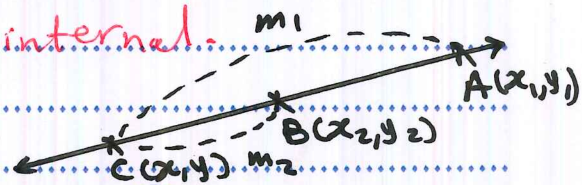
$\vec{A}+\vec{B}=\vec{C}$

- Division of a line segment :



$$C(x,y)=\frac{m_1(x_2,y_2)+m_2(x_1,y_1)}{m_1+m_2}$$

↳ internal.



$$C(x,y)=\frac{m_1(x_2,y_2)-m_2(x_1,y_1)}{m_1-m_2}$$

- Slope of st. Line.

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{-\text{coff } x}{\text{coff } y}=\frac{b}{a}$$

↳ $y=mx+c$ ↳ Two Points ↳ $ax+by=c$ ↳ $\vec{u}=(a,b)$

$=\tan\theta$

↳ angle wi The +ve direction x-axis.

- IF $L_1 \parallel L_2$ ∴ $m_1=m_2$ • IF $L_1 \perp L_2$ ∴ $m_1 \times m_2=-1$
- IF $L \parallel$ x-axis ∴ $m=0$ • IF $L \parallel$ y-axis ∴ m undefind
- Cut with x-axis put $y=0$
- Cut with y-axis put $x=0$

- equation of st. line

$$\begin{cases} \vec{r}=\vec{A}+k\vec{u} & \text{(vector)} \\ x=x_1+ak,\ y=y_1+bk & \text{(Parametric)} \\ \dfrac{y-y_1}{x-x_1}=m & \text{(Cartesian)} \end{cases}$$

$$\frac{x}{\boxed{a}}+\frac{y}{\boxed{b}}=1$$

Cut x-axis ↲ ↳ Cut y-axis.

- The general equation of any two st. line L_1, L_2

$$a_1x+b_1y+c_1+k(a_2x+b_2y+c_2)=0.$$

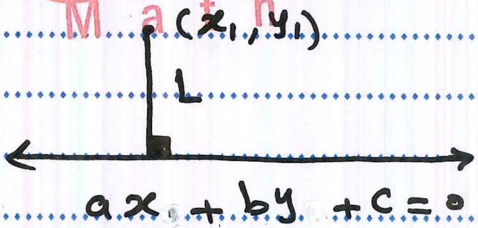
- The angle between two st. line

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| \quad \text{where } \theta \in \left[0, \frac{\pi}{2}\right]$$

m_1 slope of L_1 , m_2 slope of L_2.

- The Length perpendicular from point to st. line.

$$L = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$



Target In Math

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First : Final revision on algebra

Remember The complex numbers

The imaginary number "i"

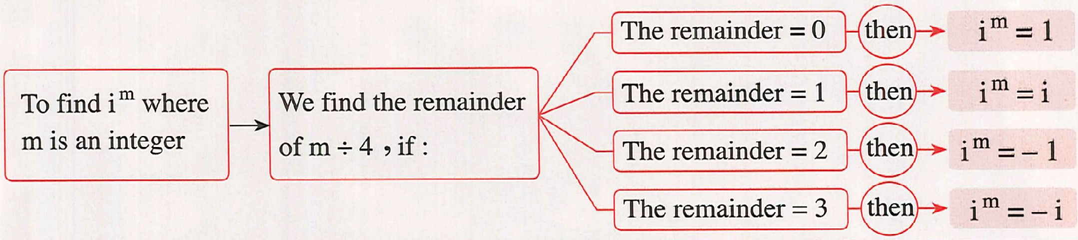
The imaginary number "i" is defined as the number whose square is – 1 $\boxed{i.e.\ i^2 = -1}$

Notice that

- $i \times i = i^2 = -1$
- $-i \times -i = i^2 = -1$

- $\sqrt{-2} = \sqrt{2\,i^2} = \sqrt{2}\,i$ Similarly :
- $\sqrt{-5} = \sqrt{5}\,i$
- $\sqrt{-9} = 3\,i$

Integer powers of "i"



For example :

- $i^{12} = 1$ "because 12 ÷ 4 = 3 and the remainder is 0"
- $i^{63} = -i$ "because 63 ÷ 4 = 15 and the remainder is 3"
- $i^{101} = i$ "because 101 ÷ 4 = 25 and the remainder is 1"
- $i^{26} = -1$ "because 26 ÷ 4 = 6 and the remainder is 2"
- i^{12n+3} "where $n \in \mathbb{Z}$" $= -i$ "because $\frac{12n+3}{4} = 3n$ and the remainder is 3"

Remark

We can express the whole one by using the imaginary number to integer powers from the multiples of the number 4 , and this helps in simplifying some imaginary numbers.

For example : • $\frac{1}{i^{19}} = \frac{i^{20}}{i^{19}} = i$ • $i^{-61} = i^{-61} \times i^{64} = i^3 = -i$

The complex number

The complex number is the number that can be written in the form : $Z = a + bi$

where a and b are two real numbers , $i^2 = -1$

Examples for complex numbers : $13 - 2i$, $7 + \sqrt{5}\,i$, -25 , $8i$, $\sqrt{15}$, $5i - 4$

Equality of two complex numbers

Two complex numbers are equal if and only if the two real parts are equal and the two imaginary parts are equal , and vice versa.

If $Z_1 = -5 + x\,i$, $Z_2 = y + \sqrt{3}\,i$ and $Z_1 = Z_2$, then $y = -5$, $x = \sqrt{3}$

Adding and subtracting complex numbers

When adding and subtracting two complex numbers , we add or subtract real parts together and add or subtract imaginary parts together.

For example :
- $(4 + 5\,i) + (-2 - 3\,i) = (4 - 2) + (5 - 3)\,i = 2 + 2\,i$
- $(26 - 4\,i) - (9 - 20\,i) = (26 - 9) + (-4 + 20)\,i = 17 + 16\,i$

Multiplying complex numbers

We use the same properties of multiplying algebraic expressions and multiplying by inspection which we have studied before.

For example :
- $2\,i\,(1 - 3\,i) = 2\,i - 6\,i^2$ (where $i^2 = -1$) $= 6 + 2\,i$
- $\underset{-10\,i}{\overset{3\,i}{(3 - 5\,i)(2 + i)}} = 6 - 7\,i - 5\,i^2$ (where $i^2 = -1$) $= 11 - 7\,i$
- $(4 - i)^2 = 16 - 8\,i + i^2$ (where $i^2 = -1$)
  $= 15 - 8\,i$
- $(5 - 3\,i)(5 + 3\,i) = 25 - 9\,i^2$ (where $i^2 = -1$)
  $= 25 + 9 = 34$

Remember that

$(a \pm b)^2 = a^2 \pm 2\,ab + b^2$

Remember that

$(a + b)(a - b) = a^2 - b^2$

The two conjugate numbers

The two numbers $a + b\,i$ and $a - b\,i$ are called conjugate numbers and we notice that the complex number and its conjugate differ only in the sign of their imaginary parts , and their sum is a real number and their product is a real number.

For example :
- The two numbers $3 + 4\,i$ and $3 - 4\,i$ are conjugate numbers , while the two numbers $2\,i - 5$ and $2\,i + 5$ are not conjugate because the imaginary part in each of them has the same sign.
- The conjugate of the number $4\,i$ is $-4\,i$
- The conjugate of the number 6 is 6

Remark

To simplify the fraction whose denominator is a complex number not real , we multiply its two terms by the conjugate of denominator.

For example : $\frac{30 + 45\,i}{1 - 2\,i} = \frac{30 + 45\,i}{1 - 2\,i} \times \frac{1 + 2\,i}{1 + 2\,i} = \frac{30 + 105\,i + 90\,i^2}{1 - 4\,i^2} = \frac{-60 + 105\,i}{5} = -12 + 21\,i$

Remember The quadratic equation in one variable (Determining the type of roots - Finding the solution set)

First Algebraic method

To determine the type of roots of the quadratic equation and find its solution set in $\mathbb{R}$ or in $\mathbb{C}$ for each of the following equations algebraically :

- $x^2 - 2x - 4 = 0$
- $4x + x^2 + 4 = 0$
- $2 + x^2 = 0$

We will follow the following steps :

Put the given equation on the form : $a x^2 + b x + c = 0$

$\therefore x^2 - 2x - 4 = 0$	$\therefore x^2 + 4x + 4 = 0$	$\therefore x^2 + 2 = 0$

Find the discriminant ($b^2 - 4ac$)

$\therefore$ The discriminant $= (-2)^2 - 4(1)(-4)$ $= 20$	$\therefore$ The discriminant $= (4)^2 - 4(1)(4)$ $= 0$	$\therefore$ The discriminant $= (0)^2 - 4(1)(2)$ $= -8$

Determining the type of roots of the equation

$\therefore$ The two roots are real and different because the discriminant is positive.	$\therefore$ The two roots are real and equal because the discriminant is zero.	$\therefore$ The two roots are complex and non real because the discriminant is negative.

Finding the solution set by factorization or the general formula

$\because x = \frac{2 \pm \sqrt{20}}{2 \times 1}$ $\therefore$ The two roots are $1 + \sqrt{5}$, $1 - \sqrt{5}$ $\therefore$ The S.S. in $\mathbb{R} = \{1 + \sqrt{5} , 1 - \sqrt{5}\}$	$\because x^2 + 4x + 4 = 0$ $\therefore (x + 2)^2 = 0$ $\therefore x = -2$ $\therefore$ The S.S. in $\mathbb{R} = \{-2\}$	$\because x^2 + 2 = 0$ $\therefore x = \pm\sqrt{-2} = \pm\sqrt{2}\, i$ $\therefore$ The S.S. in $\mathbb{R} = \varnothing$, the S.S. in $\mathbb{C} = \{\sqrt{2}\, i , -\sqrt{2}\, i\}$

Second Graphic method

To determine the type of roots of the quadratic equation and find the solution set for each of the following equations graphically :

- $x^2 - 2x - 3 = 0$
- $9 + x^2 - 6x = 0$
- $-x^2 + 3x - 5 = 0$

We will follow the following steps :

Put the given equation on the form : $a x^2 + b x + c = 0$

$\therefore x^2 - 2x - 3 = 0$

$\therefore x^2 - 6x + 9 = 0$

$\therefore -x^2 + 3x - 5 = 0$

Write the quadratic function *f* which is related by the equation

$\therefore f(x) = x^2 - 2x - 3$

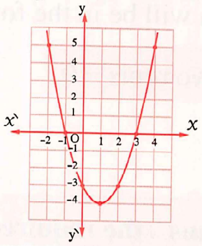
$\therefore f(x) = x^2 - 6x + 9$

$\therefore f(x) = -x^2 + 3x - 5$

Draw the curve of the function in a suitable interval from real numbers where $\frac{-b}{2a}$ is in its middle

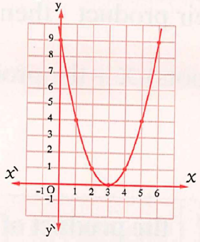
$\because \frac{-b}{2a} = \frac{2}{2} = 1$

$\therefore [-2, 4]$ is a suitable interval.



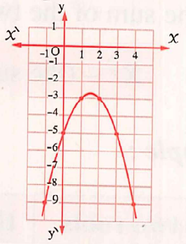
$\because \frac{-b}{2a} = \frac{6}{2} = 3$

$\therefore [0, 6]$ is a suitable interval.



$\because \frac{-b}{2a} = \frac{-3}{-2} = 1\frac{1}{2}$

$\therefore [-1, 4]$ is a suitable interval.



Determining the type of roots of the equation

The two roots are real and different because the curve intersects x-axis at two points.

The two roots are real and equal because the curve touches x-axis

The two roots are complex and non real because the curve does not intersect x-axis.

Finding the solution set in $\mathbb{R}$

The S.S. in $\mathbb{R} = \{-1, 3\}$

The S.S. in $\mathbb{R} = \{3\}$

The S.S. in $\mathbb{R} = \varnothing$

The relation between the two roots of the equation : $a x^2 + b x + c = 0$ and the coefficients of its terms

The sum of the two roots $= \frac{-b}{a}$

The product of the two roots $= \frac{c}{a}$

For example :

Equation of second degree	The sum of the two roots	The product of the two roots
• $2x^2 + 5x - 4 = 0$	$\frac{-5}{2} = -2.5$	$\frac{-4}{2} = -2$
• $3x^2 - 7x + 3 = 0$	$\frac{7}{3}$	$\frac{3}{3} = 1$ (One of the roots is the multiplicative inverse of the other)
• $5x^2 - 7 = 0$	Zero (One of the roots is the additive inverse of the other)	$\frac{-7}{5}$

Remember Forming the quadratic equation

First Forming the quadratic equation whose two roots are known

We find the sum of the two roots and their product , then the equation will be in the form :

$$x^2 - (\text{the sum of the two roots})\, x + \text{the product of the two roots} = 0$$

For example :

If the two roots are	then the sum of the two roots is	the product of the two roots is	Thus , the required equation is
• $3 , -4$	-1	-12	$x^2 + x - 12 = 0$
• $\frac{2}{3} , \frac{3}{2}$	$\frac{13}{6}$	1	$x^2 - \frac{13}{6}x + 1 = 0$ *i.e.* $6x^2 - 13x + 6 = 0$
• $2 + i , 2 - i$	4	5	$x^2 - 4x + 5 = 0$

Second Forming a quadratic equation from another given quadratic equation

First method

This method is used if finding the two roots of the given equation is easy.

For example :

If L and M are the two roots of the equation : $x^2 - x - 6 = 0$ where $L > M$

, form the quadratic equation whose roots are : $L - 2$, $M^2 + 1$

① **We find the two roots of the given equation L and M :**

$\because x^2 - x - 6 = 0 \qquad \therefore (x - 3)(x + 2) = 0$

$\therefore L = 3 \text{ , } M = -2$

② **We find the two roots of the required equation D and E :**

- $D = L - 2 = 3 - 2 = 1$
- $E = M^2 + 1 = (-2)^2 + 1 = 5$

③ **We form the required equation :**

$\therefore x^2 - 6x + 5 = 0$

Second method

This method is used if we can find "D + E" , "DE" of the required equation in terms of "L + M" , "LM" of the given equation by one of the following identities :

① $L^2 + M^2 = (L + M)^2 - 2\,LM$

② $(L - M)^2 = (L + M)^2 - 4\,LM$

③ $L^3 + M^3 = (L + M)\left[(L + M)^2 - 3\,LM\right]$

④ $L^3 - M^3 = (L - M)\left[(L + M)^2 - LM\right]$

⑤ $\frac{1}{M} + \frac{1}{L} = \frac{L + M}{LM}$

⑥ $\frac{L}{M} + \frac{M}{L} = \frac{L^2 + M^2}{LM} = \frac{(L + M)^2 - 2\,LM}{LM}$

For example :

If L and M are the two roots of the equation : $x^2 - 3x + 1 = 0$

, form the equation whose roots are : $D = \frac{L}{M}$, $E = \frac{M}{L}$

Remember The sign of the function

The sign of the constant function

The sign of the constant function $f : f(x) = c$, $c \in \mathbb{R}^*$ is the same sign of c for all values of $x \in \mathbb{R}$

For example :

- The sign of the function $f : f(x) = -7$ is negative for all values of $x \in \mathbb{R}$
- The sign of the function $f : f(x) = 2$ is positive for all values of $x \in \mathbb{R}$

The sign of the first degree function (linear function)

To determine the sign of the linear function $f : f(x) = b\,x + c$, $b \neq 0$, we put $f(x) = 0$ $\therefore b\,x + c = 0$ $\therefore x = \frac{-c}{b}$

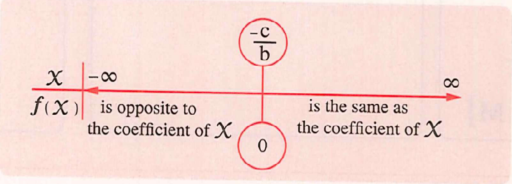
Then the sign of the function f :

① Is the same sign of b at $x > \frac{-c}{b}$

② Is opposite to the sign of b at $x < \frac{-c}{b}$

③ $f(x) = 0$ at $x = \frac{-c}{b}$

And we illustrate this on the number line as in the figure :



For example :

If $f : f(x) = -3\,x + 6$ Put $-3\,x + 6 = 0$ $\therefore x = 2$

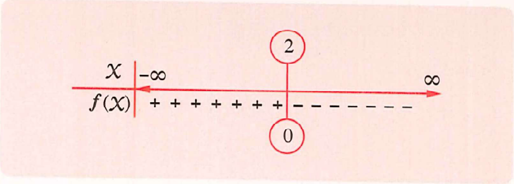
The sign of the function f :

① Is negative at $x > 2$

② Is positive at $x < 2$

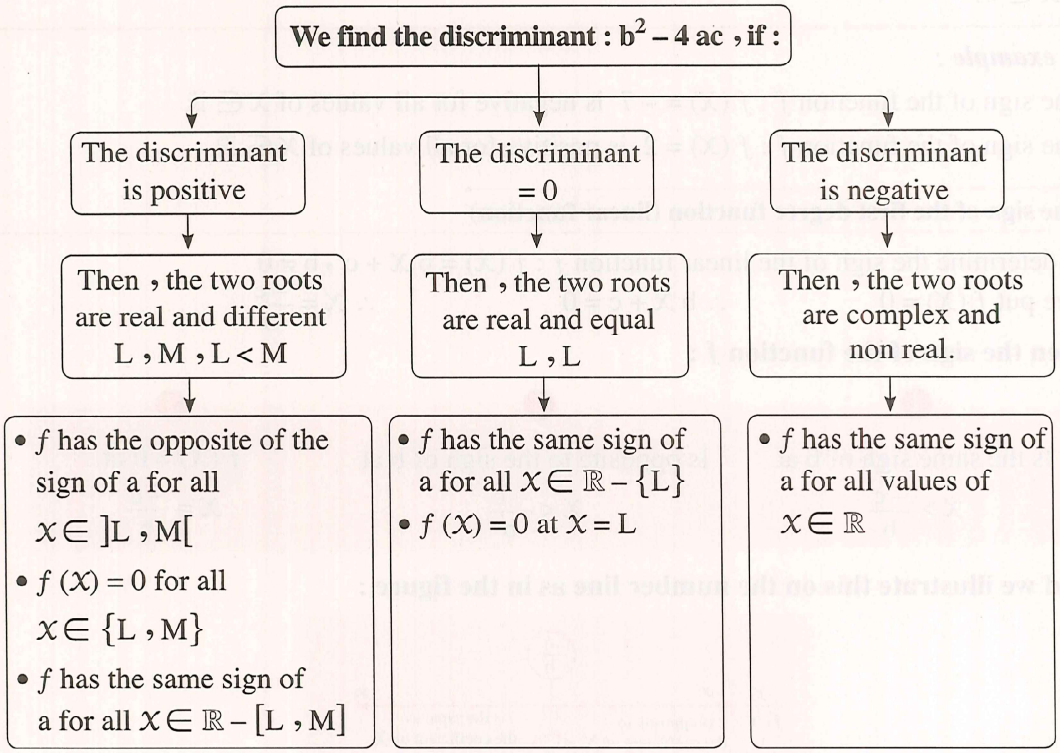
③ $f(x) = 0$ at $x = 2$

And we illustrate this on the number line as in the figure :



The sign of the second degree function (quadratic function)

To determine the sign of the quadratic function $f : f(X) = a X^2 + b X + c , a \neq 0$, we write the quadratic equation : $a X^2 + b X + c = 0$ which is related by the function , then do the following steps :



We find the discriminant : $b^2 - 4ac$, if :

The discriminant is positive

Then , the two roots are real and different L , M , L < M

- f has the opposite of the sign of a for all $X \in]L , M[$
- $f(X) = 0$ for all $X \in \{L , M\}$
- f has the same sign of a for all $X \in \mathbb{R} - [L , M]$

The discriminant = 0

Then , the two roots are real and equal L , L

- f has the same sign of a for all $X \in \mathbb{R} - \{L\}$
- $f(X) = 0$ at $X = L$

The discriminant is negative

Then , the two roots are complex and non real.

- f has the same sign of a for all values of $X \in \mathbb{R}$

For example :

If : • $f : f(X) = X^2 - 4X + 3$

• $f : f(X) = -X^2 - 2X - 1$

• $f : f(X) = 2X^2 - 3X + 5$

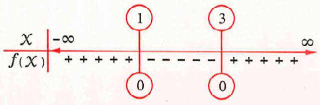
, then we can determine the sign of each of the previous functions as the following :

We write the quadratic equations which are related by the previous functions and complete the steps as follows :

$x^2 - 4x + 3 = 0$

∵ The discriminant $= (-4)^2 - 4 \times 1 \times 3$ $= 4$ (positive)

∴ The two roots are real and different and they are 3 and 1

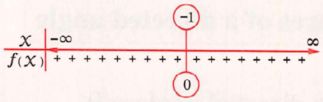


- f is negative for all $x \in]1 , 3[$
- $f(x) = 0$ for all $x \in \{1 , 3\}$
- f is positive for all $x \in \mathbb{R} - [1 , 3]$

$x^2 + 2x + 1 = 0$

∵ The discriminant $= (2)^2 - 4 \times 1 \times 1 = 0$

∴ The two roots are real and equal and each of them equals -1

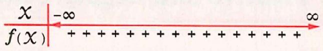


- f is positive for all $x \in \mathbb{R} - \{-1\}$
- $f(x) = 0$ at $x = -1$

$2x^2 - 3x + 5 = 0$

∵ The discriminant $= (-3)^2 - 4 \times 2 \times 5$ $= -31$ (negative)

∴ The two roots are complex and non real



- f is positive for all values of $x \in \mathbb{R}$

Remember the solving of the quadratic inequalities in $\mathbb{R}$

To find the solution set of the inequality : $x^2 - 5x + 6 > 0$ in $\mathbb{R}$:

① We write the quadratic function related by the inequality. → ② We study the sign of the quadratic function which we wrote. → ③ We determine the intervals which satisfy the inequality.

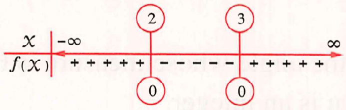
$f : f(x) = x^2 - 5x + 6$

∵ The discriminant $= (-5)^2 - 4 \times 1 \times 6$ $= 1$ (positive)

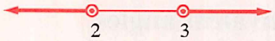
∴ The two roots are real and different

, ∵ $(x - 2)(x - 3) = 0$

∴ $x = 2$ or $x = 3$



The solution set of the inequality : $x^2 - 5x + 6 > 0$ is $\mathbb{R} - [2 , 3]$



Second : Final revision on trigonometry

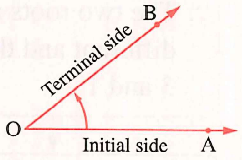
Remember The directed angle

Definition of the directed angle

The directed angle is an ordered pair of two rays called the sides of the angle with a common starting point called the vertex.

For example :

The ordered pair $\left(\overrightarrow{OA}, \overrightarrow{OB}\right)$ represents the directed angle $\angle AOB$ whose initial side is $\overrightarrow{OA}$ and terminal side is $\overrightarrow{OB}$



Positive and negative measures of a directed angle

If the positive measure of the directed angle = θ

, then the negative measure of the same directed angle = $\theta - 360°$

For example :

The negative measure of the directed angle of measure 210° = 210° – 360° = – 150°

If the negative measure of the directed angle = $-\theta$

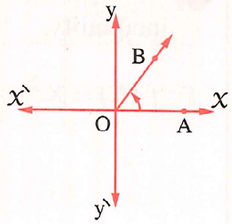
, then the positive measure of the same directed angle = $-\theta + 360°$

For example :

The positive measure of the directed angle of measure (– 120°) = – 120° + 360° = 240°

The standard position of the directed angle

A directed angle is in the standard position if the following two conditions are satisfied :



① Its initial side lies on the positive direction of the X-axis.

② Its vertex is the origin point of an orthogonal coordinate plane.

Equivalent angles

Several directed angles in the standard position are said to be equivalent when they have one common terminal side.

And we get equivalent angles to the angle whose measure is θ by adding n 360° to it or subtracting n 360° from it where n is an integer.

Determining the quadrant in which the terminal side of the directed angle ∠ AOB whose measure is θ in the standard position lies :

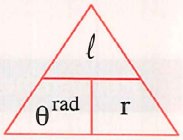
We find α the measure of the smallest positive equivalent angle to the angle θ , then if :

- $\alpha \in]0° , 90°[$ then → ∠ AOB lies in the first quadrant.
- $\alpha \in]90° , 180°[$ then → ∠ AOB lies in the second quadrant.
- $\alpha \in]180° , 270°[$ then → ∠ AOB lies in the third quadrant.
- $\alpha \in]270° , 360°[$ then → ∠ AOB lies in the fourth quadrant.
- $\alpha \in \{0° , 90° , 180° , 270° , 360°\}$ then → ∠ AOB is called a quadrantal angle.

Radian measure and degree measure of an angle

- The radian measure of a central angle in a circle = $\dfrac{\text{Length of the arc which the central angle subtends}}{\text{Length of the radius of this circle}}$

i.e. $\boxed{\theta^{rad} = \dfrac{\ell}{r}}$ and from it $\boxed{\ell = \theta^{rad}\, r}$, $\boxed{r = \dfrac{\ell}{\theta^{rad}}}$



- **The relation between the radian measure and the degree measure :**

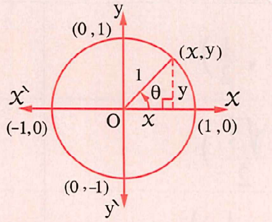
$\boxed{\dfrac{x^{\circ}}{180°} = \dfrac{\theta^{rad}}{\pi}}$ and from it $\boxed{\theta^{rad} = x^{\circ} \times \dfrac{\pi}{180°}}$, $\boxed{x^{\circ} = \theta^{rad} \times \dfrac{180^{\circ}}{\pi}}$

Notice that

π in radians is equivalent to 180° in degrees.

Remember The trigonometric functions of an acute angle and their reciprocals

- $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = y$
- $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = x$
- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{y}{x}$



$\boxed{x^2 + y^2 = 1}$

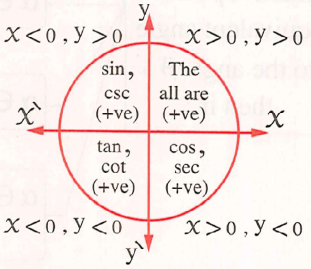
- $\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}} = \dfrac{1}{y}$
- $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \dfrac{1}{x}$
- $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{x}{y}$

Notice that

- $x \in [-1 , 1]$ and from it $\cos\theta \in [-1 , 1]$
- $y \in [-1 , 1]$ and from it $\sin\theta \in [-1 , 1]$
- The equivalent angles have the same trigonometric functions.

Remember The signs of trigonometric functions

Quadrant	The interval that θ belongs to	sign of cos , sec	sign of sin , csc	sign of tan , cot
First	$]0 , \frac{\pi}{2}[$	+	+	+
Second	$]\frac{\pi}{2} , \pi[$	−	+	−
Third	$]\pi , \frac{3\pi}{2}[$	−	−	+
Fourth	$]\frac{3\pi}{2} , 2\pi[$	+	−	−



Notice that

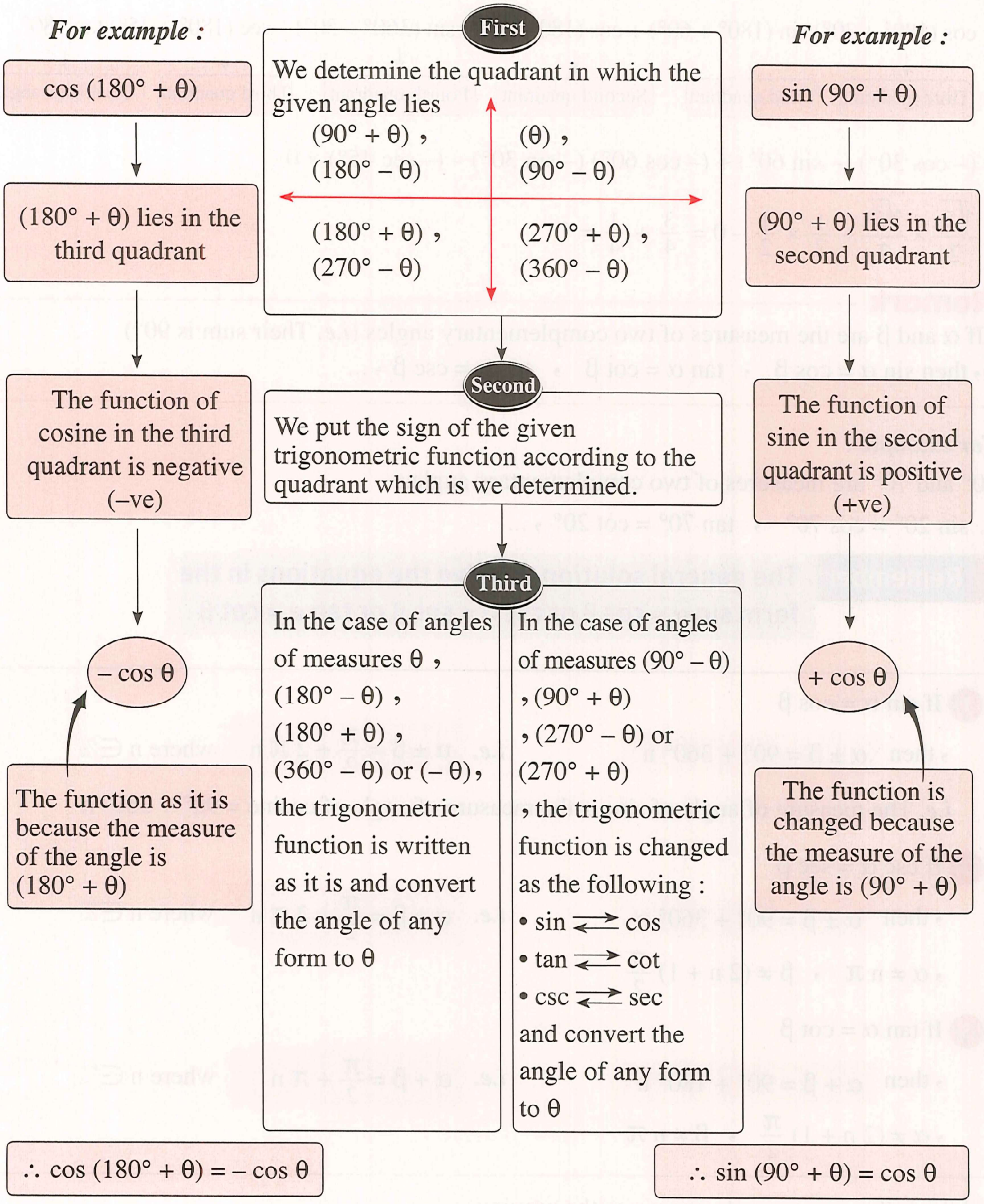
The trigonometric functions of the equivalent angles have the same sign.

Remember The trigonometric functions of some special angles

The measure of θ	The point of the intersection of the terminal side with the unit circle	The values of the trigonometric functions		
		sin θ	cos θ	tan θ
0° or 360°	(1 , 0)	0	1	0
90°	(0 , 1)	1	0	undefined
180°	(−1 , 0)	0	− 1	0
270°	(0 , − 1)	− 1	0	undefined
30°	$(\frac{\sqrt{3}}{2} , \frac{1}{2})$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
60°	$(\frac{1}{2} , \frac{\sqrt{3}}{2})$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
45°	$(\frac{1}{\sqrt{2}} , \frac{1}{\sqrt{2}})$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1

Remember The relation between the trigonometric functions of two related angles

To know how to find the relations between the trigonometric functions of two related angles , we will follow the following steps :



First

We determine the quadrant in which the given angle lies

Second

We put the sign of the given trigonometric function according to the quadrant which is we determined.

Third

In the case of angles of measures θ , (180° – θ) , (180° + θ) , (360° – θ) or (– θ) , the trigonometric function is written as it is and convert the angle of any form to θ

In the case of angles of measures (90° – θ) , (90° + θ) , (270° – θ) or (270° + θ) , the trigonometric function is changed as the following :

- sin $\rightleftarrows$ cos
- tan $\rightleftarrows$ cot
- csc $\rightleftarrows$ sec

and convert the angle of any form to θ

For example :

cos (180° + θ)

(180° + θ) lies in the third quadrant

The function of cosine in the third quadrant is negative (–ve)

The function as it is because the measure of the angle is (180° + θ)

– cos θ

∴ cos (180° + θ) = – cos θ

For example :

sin (90° + θ)

(90° + θ) lies in the second quadrant

The function of sine in the second quadrant is positive (+ve)

The function is changed because the measure of the angle is (90° + θ)

+ cos θ

∴ sin (90° + θ) = cos θ

For example :

Without using calculator, we can find :

$$\cos(-150°)\sin 600° + \cos\frac{2\pi}{3}\sin 330° - \sec\left(-\frac{5\pi}{4}\right)\tan 900°$$

$$= \cos(210°)\sin(360° + 240°) + \cos 120°\sin(360° - 30°) - \sec 225°\tan(180° + 2 \times 360°)$$

$$= \underbrace{\cos(180° + 30°)}_{\text{Third quadrant}}\underbrace{\sin(180° + 60°)}_{\text{Third quadrant}} + \underbrace{\cos(180° - 60°)}_{\text{Second quadrant}}\underbrace{\sin(360° - 30°)}_{\text{Fourth quadrant}} - \underbrace{\sec(180° + 45°)}_{\text{Third quadrant}}\underbrace{\tan 180°}_{\text{Quadrantal angle}}$$

$$= (-\cos 30°)(-\sin 60°) + (-\cos 60°)(-\sin 30°) - (-\sec 45°) \times 0$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} - 0 = \frac{3}{4} + \frac{1}{4} = 1$$

Remark

If α and β are the measures of two complementary angles (*i.e.* Their sum is 90°)

, then $\sin\alpha = \cos\beta$, $\tan\alpha = \cot\beta$, $\sec\alpha = \csc\beta$, ...

For example :

20° and 70° are measures of two complementary angles.

$\therefore \sin 20° = \cos 70°$, $\tan 70° = \cot 20°$, ...

Remember The general solution to solve the equations in the form $\sin\alpha = \cos\beta$ or $\csc\alpha = \sec\beta$ or $\tan\alpha = \cot\beta$

① If $\sin\alpha = \cos\beta$

, then $\alpha \pm \beta = 90° + 360°\ n$ *i.e.* $\alpha \pm \beta = \frac{\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$

i.e. The measure of angle of sine ± the measure of angle of cosine = 90° + 360° n

② If $\csc\alpha = \sec\beta$

, then $\alpha \pm \beta = 90° + 360°\ n$ *i.e.* $\alpha \pm \beta = \frac{\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$

, $\alpha \neq n\pi$, $\beta \neq (2n + 1)\frac{\pi}{2}$

③ If $\tan\alpha = \cot\beta$

, then $\alpha + \beta = 90° + 180°\ n$ *i.e.* $\alpha + \beta = \frac{\pi}{2} + \pi n$ where $n \in \mathbb{Z}$

, $\alpha \neq (2n + 1)\frac{\pi}{2}$, $\beta \neq n\pi$

and the following example expresses the previous :

- If $\sin 4\theta = \cos 2\theta \text{ , } \theta \in \left]0 \text{ , } \frac{\pi}{2}\right[$

$$\therefore 4\theta \pm 2\theta = \frac{\pi}{2} + 2\pi n \text{ , } n \in \mathbb{Z}$$

Or

$\therefore 2\theta = \frac{\pi}{2} + 2\pi n$

$\therefore \theta = \frac{\pi}{4} + \pi n$

- At $n = 0$

$\therefore \theta = \frac{\pi}{4} = 45°$

- At $n = 1$

$\therefore \theta = \frac{\pi}{4} + \pi$

(refused)

$6\theta = \frac{\pi}{2} + 2\pi n$

$\theta = \frac{\pi}{12} + \frac{\pi}{3} n$

- At $n = 0$

$\therefore \theta = \frac{\pi}{12} = 15°$

- At $n = 1$

$\therefore \theta = \frac{\pi}{12} + \frac{\pi}{3} = 75°$

- At $n = 2$

$\therefore \theta = \frac{\pi}{12} + \frac{2\pi}{3}$

(refused)

$$\therefore \theta = 15° \text{ , } 45° \text{ or } 75°$$

- If $\tan 3\theta = \cot 2\theta \text{ , } \theta \in \left]0 \text{ , } \frac{\pi}{2}\right[$

$$\therefore 3\theta + 2\theta = \frac{\pi}{2} + \pi n \text{ , } n \in \mathbb{Z}$$

$\therefore 5\theta = \frac{\pi}{2} + \pi n$

$\therefore \theta = \frac{\pi}{10} + \frac{\pi}{5} n$

- At $n = 0$

$\therefore \theta = \frac{\pi}{10} = 18°$

- At $n = 1$

$\therefore \theta = \frac{\pi}{10} + \frac{\pi}{5} = \frac{3\pi}{10} = 54°$

- At $n = 2$

$\therefore \theta = \frac{\pi}{10} + \frac{2\pi}{5} = \frac{1}{2}\pi$

(refused)

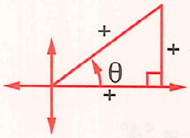
$$\therefore \theta = 18° \text{ or } 54°$$

Remember How to find the measure of an angle (θ) given the value of one of its trigonometric ratios (a)

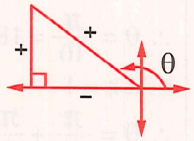
Examples Steps	$\sin \theta = -\frac{1}{2}$	$\cos \theta = \frac{1}{\sqrt{2}}$	$\tan \theta = -\sqrt{3}$
① We determine the quadrant in which θ lies according to the sign of a	The sine function is negative. ∴ θ lies in the third or the fourth quadrant.	The cosine function is positive. ∴ θ lies in the first or the fourth quadrant.	The tangent function is negative. ∴ θ lies in the second or the fourth quadrant
② We find the measure of the acute angle α whose trigonometric function = | a |	$\sin \alpha = \left\| -\frac{1}{2} \right\| = \frac{1}{2}$ $\therefore \alpha = 30°$	$\cos \alpha = \left\| \frac{1}{\sqrt{2}} \right\| = \frac{1}{\sqrt{2}}$ $\therefore \alpha = 45°$	$\tan \alpha = \left\| -\sqrt{3} \right\| = \sqrt{3}$ $\therefore \alpha = 60°$
③ We put the angle θ in the quadrant that we determined at the first step by using one of the relations : 180° – α , 180° + α or 360° – α	∵ θ lies in the third quadrant. $\therefore \theta = 180° + \alpha = 180° + 30° = 210°$ or θ lies in the fourth quadrant. $\therefore \theta = 360° - \alpha = 360° - 30° = 330°$	∵ θ lies in the first quadrant. $\therefore \theta = \alpha = 45°$ or θ lies in the fourth quadrant $\therefore \theta = 360° - \alpha = 360° - 45° = 315°$	∵ θ lies in the second quadrant. $\therefore \theta = 180° - \alpha = 180° - 60° = 120°$ or θ lies in the fourth quadrant $\therefore \theta = 360° - \alpha = 360° - 60° = 300°$

Remember How to find all the trigonometric functions of an angle given the value of one of its trigonometric functions

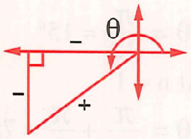
We can find the values of the trigonometric functions of an angle directly if we draw the angle in its standard position and we draw the right-angled triangle that represents it by using the value of the given trigonometric function concerning the signs according to the quadrant in which the angle lies as follows :



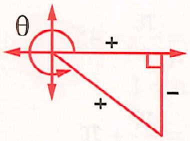
In the 1st quadrant



In the 2nd quadrant

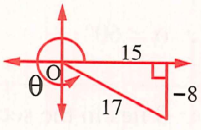
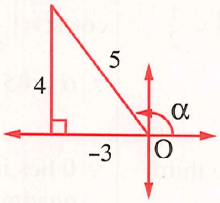
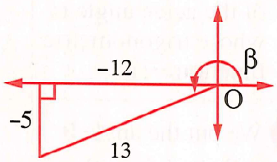


In the 3rd quadrant



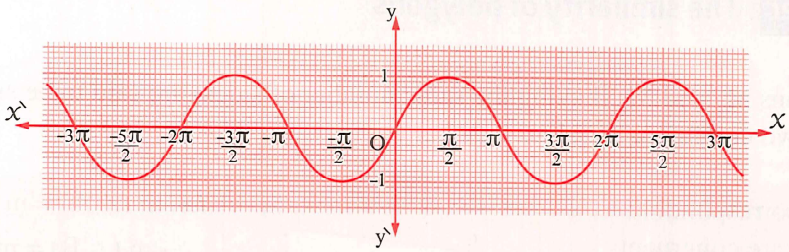
In the 4th quadrant

For example :

$\sin\theta = \frac{-8}{17}$ where $270° < \theta < 360°$	$\cos\alpha = \frac{-3}{5}$ where α is the smallest positive angle.	$\tan\beta = \frac{5}{12}$ where β is the greatest positive angle , $0° < \beta < 360°$
$\because 270° < \theta < 360°$ $\therefore \theta$ lies in the fourth quadrant. 	$\because \cos\alpha$ is negative $\therefore \alpha$ lies in the second or the third quadrant , $\because \alpha$ is the smallest positive angle. $\therefore \alpha$ lies in the second quadrant. 	$\because \tan\beta$ is positive $\therefore \beta$ lies in the first or the third quadrant , $\because \beta$ is the greatest positive angle. $\therefore \beta$ lies in the third quadrant 
$\therefore \cos\theta = \frac{15}{17}$, $\tan\theta = \frac{-8}{15}$, ...	$\therefore \sin\alpha = \frac{4}{5}$, $\tan\alpha = \frac{-4}{3}$, ...	$\therefore \sin\beta = \frac{-5}{13}$, $\cos\beta = \frac{-12}{13}$, ...

Remember The properties of the sine function and the cosine function

Properties of the sine function $f : f(\theta) = \sin\theta$



① The domain of the sine function is $]-\infty , \infty[$

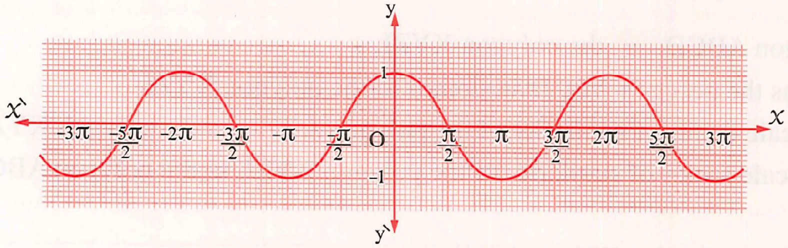
② • The maximum value of the function is 1 and it happens when $\theta = \frac{\pi}{2} + 2n\pi , n \in \mathbb{Z}$

• The minimum value of the function is – 1 and it happens when $\theta = \frac{3\pi}{2} + 2n\pi , n \in \mathbb{Z}$

③ The range of the function = $[-1 , 1]$

④ The function is periodic and its period is 2π (360°)

Properties of the cosine function $f : f(\theta) = \cos\theta$



① The domain of the cosine function is $]-\infty , \infty[$

② • The maximum value of the function is 1 and it happens when $\theta = \pm 2n\pi , n \in \mathbb{Z}$

• The minimum value of the function is – 1 and it happens when $\theta = \pi \pm 2\pi n , n \in \mathbb{Z}$

③ The range of the function = $[-1 , 1]$

④ The function is periodic and its period is 2π (360°)

Remark

Each of the two functions $f : f(\theta) = a \sin b\theta$, $f : f(\theta) = a \cos b\theta$ is periodic , its period is $\frac{2\pi}{|b|}$ and its range is $[-a , a]$ where a is positive.

For example : • $f : f(\theta) = 5 \sin\theta$ its period is 2π and its range is $[-5 , 5]$

• $f : f(\theta) = 3 \cos 7\theta$ its period is $\frac{2\pi}{7}$ and its range is $[-3 , 3]$

Third : Final revision on geometry

Remember The similarity of polygons

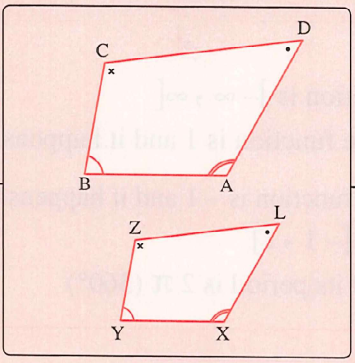
Two polygons M_1 and M_2 (having the same number of sides) are said to be similar if the following two conditions satisfied together :

① Their corresponding angles are congruent.

i.e. $m(\angle A) = m(\angle X)$
$, m(\angle B) = m(\angle Y)$
$, m(\angle C) = m(\angle Z)$
$, m(\angle D) = m(\angle L)$

② The lengths of their corresponding sides are proportional.

i.e. $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CD}{ZL} = \frac{DA}{LX} = K$



In this case , we say that :

- The polygon ABCD $\sim$ the polygon XYZL ,
  that means the polygon ABCD is similar to the polygon XYZL
- K is the scale factor of similarity of the polygon ABCD to the polygon XYZL
- $\frac{1}{K}$ is the scale factor of similarity of the polygon XYZL to the polygon ABCD

Remarks

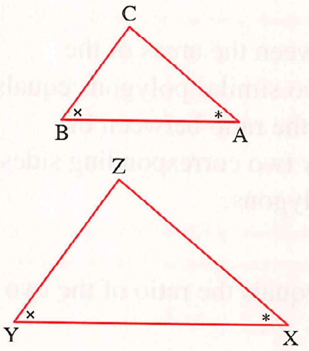
- On writing the similar polygons , write them according to the order of their corresponding vertices.
- If each one of two polygons is similar to a third polygon , then the two polygons are similar.
- All regular polygons which have the same number of sides are similar
  (All equilateral triangles are similar , all squares are similar , all regular pentagons are similar , ...)
- If K is the similarity ratio of polygon M_1 to polygon M_2 , and :
  If $K > 1$, then polygon M_1 is an enlargement of polygon M_2 , where K is called the enlargement ratio.
  If $0 < K < 1$, then polygon M_1 is a shrinking to polygon M_2 , where K is called the shrinking ratio.
  If $K = 1$, then polygon M_1 is congruent to polygon M_2
- The ratio between the perimeters of two similar polygons = the ratio between the lengths of two corresponding sides of them.

Remember The similarity of triangles

Two triangles are similar

First case

If two angles of one triangle are congruent to their corresponding angles of the other triangle.



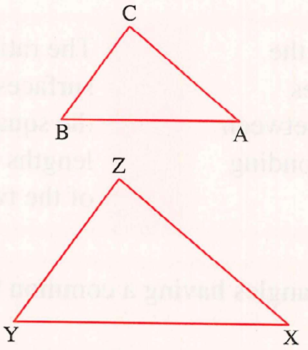
If $\angle A \equiv \angle X$

, $\angle B \equiv \angle Y$

, then $\Delta ABC \sim \Delta XYZ$

Second case

If the side lengths of two triangles are in proportion.

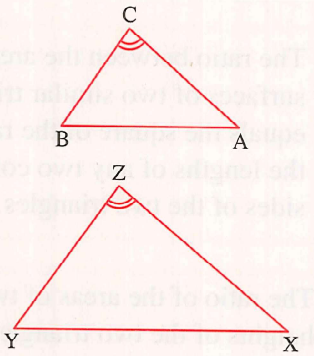


If $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}$

, then $\Delta ABC \sim \Delta XYZ$

Third case

If an angle of one triangle is congruent to an angle of the other triangle and the lengths of the sides including those angles are in proportion.



If $\angle C \equiv \angle Z$

, $\frac{CA}{ZX} = \frac{CB}{ZY}$

, then $\Delta ABC \sim \Delta XYZ$

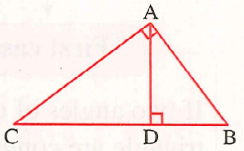
Remarks

- Two isosceles triangles are similar if the measure of an angle in one of them is equal to the measure of the corresponding angle in the other triangle.
- Two right-angled triangles are similar if the measure of an acute angle in one of them is equal to the measure of an acute angle in the other triangle.

Corollary

In any right-angled triangle , the altitude to the hypotenuse separates the triangle into two triangles which are similar to each other and to the original triangle.

In the opposite figure :



If ΔABC is a right-angled triangle at A and $\overline{AD} \perp \overline{BC}$

, then $\Delta DBA \sim \Delta DAC \sim \Delta ABC$ and from this we can deduce that :

- $(AB)^2 = BD \times BC$
- $(AC)^2 = CD \times CB$
- $(AD)^2 = BD \times DC$
- $AD \times BC = AB \times AC$

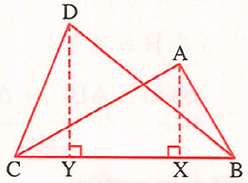
Remember The relation between the areas of two similar polygons

The ratio between the areas of the surfaces of two similar triangles equals the square of the ratio between the lengths of any two corresponding sides of the two triangles.

The ratio between the areas of the surfaces of two similar polygons equals the square of the ratio between the lengths of any two corresponding sides of the two polygons.

The ratio of the areas of two triangles having a common base equals the ratio of the two heights of the two triangles.

In the opposite figure :



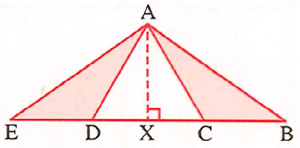
$\overline{BC}$ is a common base of $\Delta\Delta$ ABC , DBC

$$\therefore \frac{a(\Delta ABC)}{a(\Delta DBC)} = \frac{\frac{1}{2} BC \times AX}{\frac{1}{2} BC \times DY} = \frac{AX}{DY}$$

Notice that : It is not necessary that the two triangles are similar.

The ratio of the areas of two triangles having a common height equals the ratio of the lengths of two bases of the two triangles.

In the opposite figure :



AX is a common height for $\Delta\Delta$ ABC , ADE

$$\therefore \frac{a(\Delta ABC)}{a(\Delta ADE)} = \frac{\frac{1}{2} BC \times AX}{\frac{1}{2} DE \times AX} = \frac{BC}{DE}$$

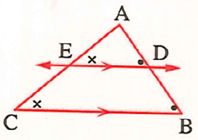
Notice that : It is not necessary that the two triangles are similar.

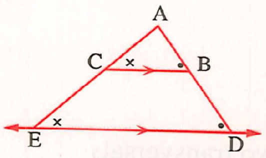
If a line is drawn parallel to one side of a triangle and intersects the other two sides or the lines containing them , then :

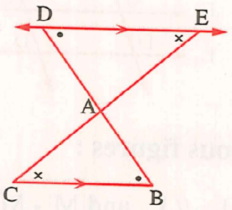
The resulting triangle is similar to the original triangle

It divides them into segments whose lengths are proportional

In each of the following figures :







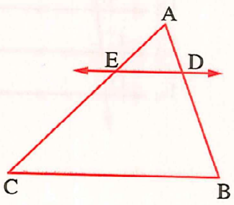
If $\overleftrightarrow{DE} \parallel \overline{BC}$ and intersects $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ at D and E respectively , then :

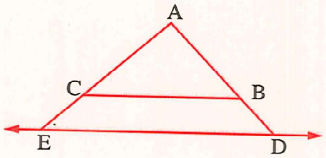
- $\Delta ADE \sim \Delta ABC$
- $\frac{AD}{DB} = \frac{AE}{EC}$ and from the properties of the proportion , we get :

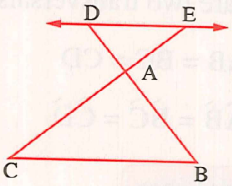
$\frac{AD}{AB} = \frac{AE}{AC}$, $\frac{AB}{DB} = \frac{AC}{CE}$

If a straight line intersects two sides of a triangle and divides them into segments whose lengths are proportional , then it is parallel to the third side of the triangle.

In each of the following figures :



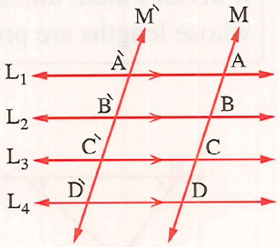


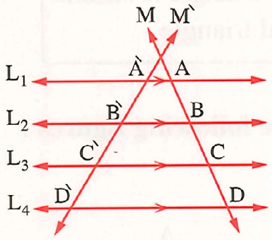


If $\frac{AD}{DB} = \frac{AE}{EC}$, then $\overleftrightarrow{DE} \parallel \overline{BC}$

Remember Talis' theorem

Given several coplanar parallel lines and two transversals , then the lengths of the corresponding segments on the transversals are proportional.





In the previous figures :

If $L_1 // L_2 // L_3 // L_4$ and M , $\grave{M}$ are two transversals

, then $\frac{AB}{\grave{A}\grave{B}} = \frac{BC}{\grave{B}\grave{C}} = \frac{CD}{\grave{C}\grave{D}} = \frac{AC}{\grave{A}\grave{C}}$.

Remember Talis' special theorem

If the lengths of the segments on the transversal are equal , then the lengths of the segments on any other transversal will be also equal.

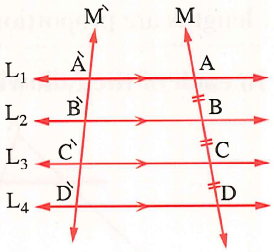
In the opposite figure :

If $L_1 // L_2 // L_3 // L_4$,

M , $\grave{M}$ are two transversals to them

and if $AB = BC = CD$

, then $\grave{A}\grave{B} = \grave{B}\grave{C} = \grave{C}\grave{D}$



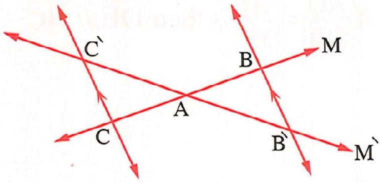
Special case

If the two lines M and $\grave{M}$ intersect at

the point A and $\overleftrightarrow{B\grave{B}} // \overleftrightarrow{C\grave{C}}$

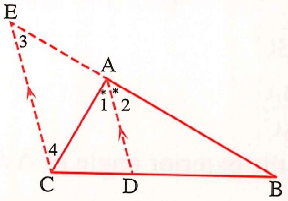
, then $\frac{AB}{AC} = \frac{A\grave{B}}{A\grave{C}}$

and conversely if $\frac{AB}{AC} = \frac{A\grave{B}}{A\grave{C}}$, then $\overleftrightarrow{B\grave{B}} // \overleftrightarrow{C\grave{C}}$



Theorem

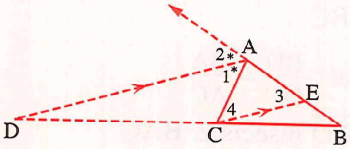
The bisector of the interior or exterior angle of a triangle at any vertex divides the opposite base of the triangle internally or externally into two parts , the ratio of their lengths is equal to the ratio of the lengths of the other two sides of the triangle.



$\because \overrightarrow{AD}$ bisects $\angle$ BAC internally.

$\therefore \boxed{\frac{BD}{DC} = \frac{AB}{AC}}$

, $\boxed{AD = \sqrt{AB \times AC - BD \times DC}}$

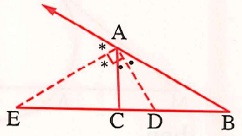


$\because \overrightarrow{AD}$ bisects $\angle$ BAC externally.

$\therefore \boxed{\frac{BD}{DC} = \frac{AB}{AC}}$

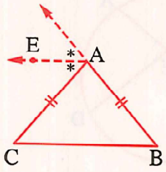
, $\boxed{AD = \sqrt{BD \times DC - AB \times AC}}$

The interior and exterior bisectors of the same angle of the triangle are perpendicular.



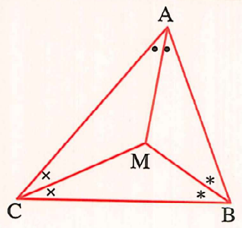
i.e. If $\overrightarrow{AD}$ and $\overrightarrow{AE}$ are the bisectors of the angle A and the exterior angle of Δ ABC at A , then $\boxed{\overline{AD} \perp \overline{AE}}$

The exterior bisector of the vertex angle of an isosceles triangle is parallel to the base.

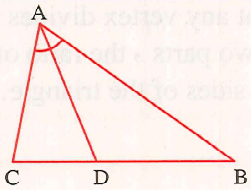


i.e. If AB = AC , $\overrightarrow{AE}$ bisects the exterior angle at A , then $\boxed{\overrightarrow{AE} \parallel \overline{BC}}$

The bisectors of angles of a triangle are concurrent.



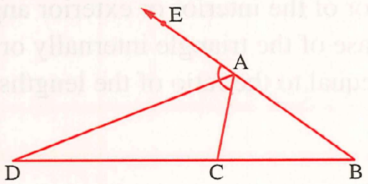
Converse of the theorem



If D $\in \overline{BC}$

such that : $\frac{BD}{DC} = \frac{BA}{AC}$

, then $\overrightarrow{AD}$ bisects $\angle$ BAC



If D $\in \overrightarrow{BC}$, D $\notin \overline{BC}$

such that : $\frac{BD}{DC} = \frac{BA}{AC}$

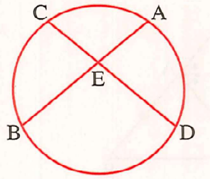
, then $\overrightarrow{AD}$ bisects the exterior angle of Δ ABC at A

Well known problem and a corollary on it

Well known problem

If $\overline{AB}$, $\overline{CD}$ are two chords in a circle

, $\overline{AB} \cap \overline{CD} = \{E\}$

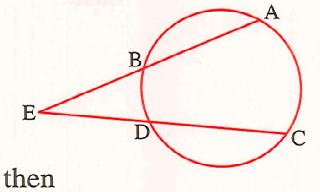


then

EA × EB = EC × ED

If $\overline{AB}$ and $\overline{CD}$ are two chords in a circle

, $\overrightarrow{AB} \cap \overrightarrow{CD} = \{E\}$

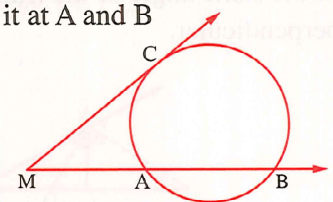


then

EA × EB = EC × ED

Corollary

If M is a point outside the circle , $\overrightarrow{MC}$ touches the circle at C , $\overrightarrow{MB}$ intersects it at A and B



then

$(MC)^2 = MA \times MB$

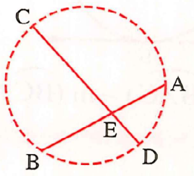
Converse of the well known problem and the corollary

Converse of the well known problem

If $\overline{AB} \cap \overline{CD} = \{E\}$,

A , B , C , D and E are distinct points and

$EA \times EB = EC \times ED$

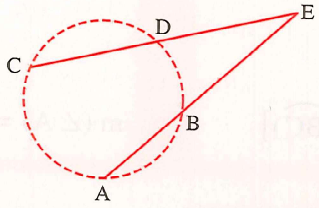


, then the points A , B , C and D lie on the same circle.

If $\overrightarrow{AB} \cap \overrightarrow{CD} = \{E\}$,

A , B , C , D and E are distinct points and

$EA \times EB = EC \times ED$

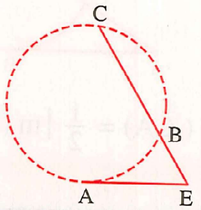


, then the points A , B , C and D lie on the same circle.

Converse of the corollary

If $E \in \overrightarrow{CB}$, $E \notin \overline{BC}$,

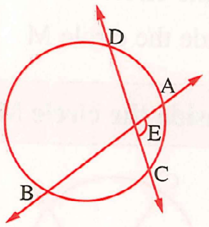
and $(EA)^2 = EB \times EC$



, then $\overline{EA}$ is a tangent segment to the circle which passes through the points A , B and C

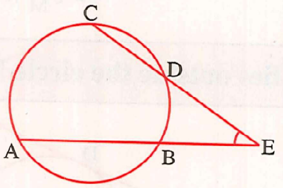
Secant , tangent and measures of angles

① The measure of an angle formed by two chords that intersect inside a circle is equal to half the sum of the measures of the intercepted arcs.



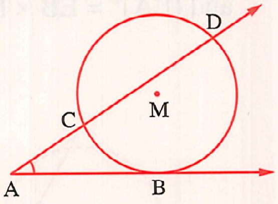
$$m (\angle AEC) = \frac{1}{2} [m (\overset{\frown}{AC}) + m (\overset{\frown}{BD})]$$

② The measure of an angle formed by two secants drawn from a point outside a circle is equal to half the positive difference of the measures of the intercepted arcs.



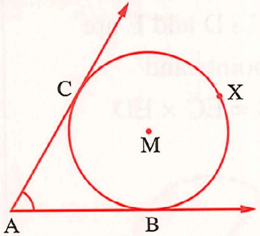
$$m (\angle E) = \frac{1}{2} [m (\overset{\frown}{AC}) - m (\overset{\frown}{BD})]$$

③ The measure of an angle formed by a secant and a tangent drawn from a point outside a circle is equal to half the positive difference of the measures of the intercepted arcs.



$$m(\angle A) = \frac{1}{2}\left[m(\overset{\frown}{BD}) - m(\overset{\frown}{BC})\right]$$

④ The measure of an angle formed by two tangents drawn from a point outside a circle is equal to half the positive difference of the measures of the intercepted arcs.

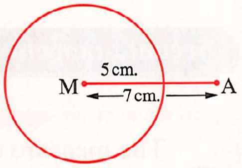


$$m(\angle A) = \frac{1}{2}\left[m(\overset{\frown}{BXC}) - m(\overset{\frown}{BC})\right]$$

Power of a point with respect to a circle

Power of the point A with respect to the circle M in which , the length of its radius r is the real number $P_M(A)$ where $P_M(A) = (AM)^2 - r^2$

For example : **In the opposite figure :**

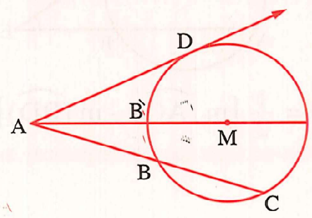
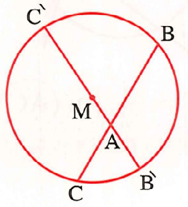


If A is a point outside the circle M

whose radius length equals 5 cm. ,

where MA = 7 cm. , then $P_M(A) = 7^2 - 5^2 = 24$

If $\rightarrow P_M(A) > 0$, then $\longrightarrow$ A lies outside the circle M
If $\rightarrow P_M(A) = 0$, then $\longrightarrow$ A lies on the circle M
If $\rightarrow P_M(A) < 0$, then $\longrightarrow$ A lies inside the circle M

If A lies outside the circle M , then :	**If A lies inside the circle M , then :**
 $P_M(A) = AB \times AC = AB' \times AC' = (AD)^2$	 $P_M(A) = -AB \times AC = -AB' \times AC'$